Connecting red cells in a bichromatic Voronoi diagram

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Abstract. Let S be a set of $n+m$ sites, of which n are red and have weight w_R, and m are blue and weigh w_B. The objective of this paper is to calculate the minimum value of w_R such that the union of the red Voronoi cells in the weighted Voronoi diagram of S is a connected set. The problem is solved for the multiplicatively-weighted Voronoi diagram in $\mathcal{O}((n+m)^2 \log(nm))$ time and for the additively-weighted Voronoi diagram in $\mathcal{O}(nm \log(nm))$ time.

Introduction

Suppose that $S = R \cup B$ is a set of $n+m$ sites, n of which are red and m of which are blue; and let $\mathrm{VD}(S)$ denote the ordinary Voronoi diagram of S. A Voronoi cell of $\mathrm{VD}(S)$ is said to be red (resp. blue) if the corresponding generator site is red (blue). The goal of this paper is to connect the red cells in order to allow one to travel within red regions following paths that do not cross blue regions. If this is not possible for $\mathrm{VD}(S)$, then there are several options to make this happen; for instance, one can add red sites or delete some blue sites, or even move sites. The approach chosen in the following is to assign different weights to red and blue sites and therefore consider their weighted Voronoi diagram (using multiplicatively- and additively-weighted distances) [**5**]. All red sites are assigned the same weight w_R, as all blue sites are assigned w_B. Consequently, the main goal is to calculate the values of w_R and w_B for which the union of red cells is connected. As it is easy to realise, in these conditions the only relevant data is the relationship between w_R and w_B. Therefore, and assuming w_B constant, the problem can be restated as calculating the minimum weight w_R^* of the red sites that connects all red cells under the appropriate diagram.

Let $\mathrm{VD}^w(S)$ denote the weighted Voronoi diagram of S. Bear in mind that $\mathrm{VD}^w(S) = \mathrm{VD}(R)$ when the weight of R tends to infinity, which assures the existence of a solution to our problem. It is clear that the structure of $\mathrm{VD}^w(S)$ depends on the weight of R

[2]Partially supported by Projects MTM2008-05043 and HP2008-0060.
[4]Supported by FCT through CIDMA of University of Aveiro.
[7]Partially supported by Projects MTM2009-07242 and Gen. Cat. DGR 2009SGR1040.
[8]Supported by FCT through CIDMA of University of Aveiro and grant SFRH/BPD/66572/2009.

and if w_R is small enough then $\mathrm{VD}^w(S) = \mathrm{VD}(B)$. Starting from this case, increasing w_R will expand the red cells and eventually two of them will meet and form one red connected component of $\mathrm{VD}^w(S)$. The point where these two cells meet is called a *critical point*. As red cells keep expanding, more will connect at different weights and each of these weights will be defined by a sole critical point. The sought weight is the one that finally unites the last two disconnected red components of $\mathrm{VD}^w(S)$. Consequently, finding these critical points is the key to solve the problem. The remainder of this paper is divided in two sections that correspond to the two types of weighted distances in question: multiplicatively- and additively-weighted distance, respectively.

1 Multiplicatively-weighted distance

The weighted distance used in this section is called the *multiplicatively-weighted distance*. Given a point p on the plane and $s_i \in S$, its distance is defined by $d_M(p, s_i) = \frac{1}{w} d_E(p, s_i)$, where d_E is the Euclidean distance and w should be replaced by the current weight of R if $s_i \in R$ or w_B if $s_i \in B$. The multiplicatively-weighted distance characterises the multiplicatively-weighted Voronoi diagram of S [**2**]. As previously mentioned, critical points are the key to calculate w_R^*, and in order to find them we need to understand how red cells form clusters on this diagram. To this end, the following definition characterises the events where the red cells of $\mathrm{VD}^w(S)$ meet. Let $b(s_i, s_j) = \{p \in \mathbb{R}^2 : d_M(p, s_i) = d_M(p, s_j)\}$ denote the bisector between sites s_i and s_j.

Definition 1.1. If w_R is the exact weight of R when two red cells of $\mathrm{VD}^w(S)$ meet for the first time at point c, then c is a *critical point of type I* if there exist red sites r_i and r_j and blue sites b_k and b_l such that $\{c\} = b(r_i, b_k) \cap b(r_i, b_l) \cap b(r_j, b_k) \cap b(r_j, b_l)$. Otherwise c is a *type II critical point:* if $w_R < w_B$ and c belongs to the blue cell of $\mathrm{VD}^w(S)$ defined by b_k then $\{c\} = b(r_i, b_k) \cap b(r_j, b_k)$; if $w_R > w_B$ and c belongs to the red cell of $\mathrm{VD}^w(S)$ defined by r_i then $\{c\} = b(r_i, b_k) \cap b(r_i, b_l)$.

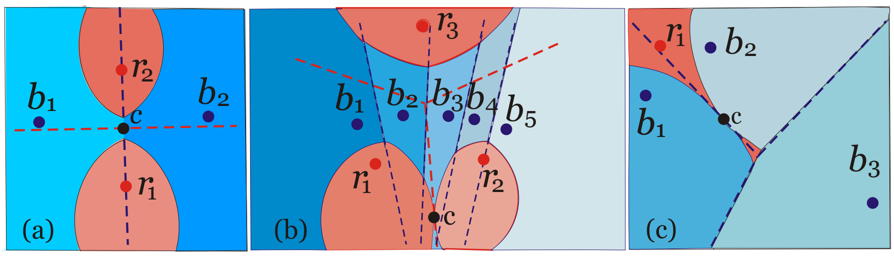

Figure 1. $\mathrm{VD}(R)$ is shown in a dashed red trace and $\mathrm{VD}(B)$ in dark blue. (a) Point c is a type I critical point. (b) Point c is a type II critical point for $w_R < w_B$. (c) Point c is a type II critical point for $w_R > w_B$.

Figure 1(a) shows clearly that a type I critical can also be found as an intersection point between $\mathrm{VD}(R)$ and $\mathrm{VD}(B)$. As the weight of R increases, the red cells defined by r_1 and r_2 will meet at c and form one red connected component of $\mathrm{VD}^w(S)$. Figures 1(b) and 1(c) illustrate two examples of critical points of type II. Figure 1(c) also shows that the regions of the multiplicatively-weighted Voronoi diagram may be disconnected. Therefore, a critical point is also created when a region meets itself (in this case, point c at the intersection of the two red cells defined by r_1).

Proposition 1.2. *Red connected components of* $\mathrm{VD}^w(S)$ *can only meet at critical points of type I or II.*

Proposition 1.3. *Red connected components of* $\mathrm{VD}^w(S)$ *can only meet* $\mathcal{O}(n+m)$ *times.*

A method to find candidates to critical points follows directly from Definition 1.1. As previously noted, candidates to type I critical points are easily found since they exist on the intersections of the edges of $\mathrm{VD}(R)$ with $\mathrm{VD}(B)$. If $\{c\} = b(r_i, r_j) \cap b(b_k, b_l)$, then the weight w_R needed to reach c with red sites is given by

$$d_M(c, r_i) = w_B \, \frac{d_E(c, r_i)}{d_E(c, b_k)}.$$

Candidates to type II critical points depend on the relationship between w_R and w_B to be found on edges of $\mathrm{VD}(R)$ or on edges of $\mathrm{VD}(B)$. However, the procedure to find them is similar and so only the first case will be described: candidates to critical points on edges of $\mathrm{VD}(R)$ that fall in the interior of cells of $\mathrm{VD}(B)$. Since an edge of $\mathrm{VD}(R)$ may cross several cells of $\mathrm{VD}(B)$ (see Figure 1(b)), one decides which is the blue site responsible for c by computing all the intersection points between such edge of $\mathrm{VD}(R)$, $b(r_i, r_j)$, and $\mathrm{VD}(B)$. For each intersection point p_k that corresponds to a blue site b_k, calculate $d_M(p_k, r_i) - d_M(p_k, b_k)$. This distance will be zero at c and therefore studying how this value alternates along $b(r_i, r_j)$ is the solution to find the cell of $\mathrm{VD}(B)$ where c is. Once blue site b_k is found, one can work out the functions that describe $b(r_i, b_k)$ and $b(r_j, b_k)$, both depending on the unknown weight of R. Finally, c is found at the minimum weight of R for which these bisectors are tangent.

Proposition 1.4. *There are* $\mathcal{O}(nm)$ *candidates to critical points.*

This proposition shows that there is a gap between the actual number of critical points and the number of candidates to critical points. To conclude, a binary search will then locate w_R^* amongst these candidates, as stated in the following theorem.

Theorem 1.5. *Concerning the multiplicatively-weighted distance, weight* w_R^* *can be found in* $\mathcal{O}((n+m)^2 \log(nm))$ *time.*

Proof. The first task to find candidates to critical points is to construct and intersect $\mathrm{VD}(B)$ and $\mathrm{VD}(R)$, which takes $\mathcal{O}(nm \log(n{+}m))$ time [**3**]. According to Proposition 1.4, there are $\mathcal{O}(nm)$ of these candidates that correspond to $\mathcal{O}(nm)$ different weights. Afterwards, sort these weights into a list in ascending order, which takes $\mathcal{O}(nm \log(nm))$ time. A binary search will locate w_R^* on this list: for each listed weight, construct $\mathrm{VD}^w(S)$ in $\mathcal{O}((n+m)^2)$ time [**2**]. Using this diagram, build a graph G that has a node for each red cell and two nodes are connected if the respective cells are neighbours. To verify if G is connected, traverse it using the Depth-First Search algorithm that runs in $\mathcal{O}((n+m)^2)$ time [**4**]. If G is indeed connected, then the search proceeds to lower weights, otherwise it proceeds to higher weights. Finally, this step takes $\mathcal{O}((n+m)^2)$ time for each weight and so it is concluded in $\mathcal{O}((n+m)^2 \log(nm))$ time. □

2 Additively-weighted distance

The weighted distance studied in this section is called the *additively-weighted distance* and is defined by $d_A(p, s_i) = d_E(p, s_i) - w$. Such distance characterises the additively-weighted Voronoi diagram of S [**1**]. The method to find critical points on this diagram is similar to the one used in the last section —the main difference here is that the regions of the additively-weighted Voronoi diagram are always connected, if they exist. Let $b(s_i, s_j) = \{p \in \mathbb{R}^2 : d_A(p, s_i) = d_A(p, s_j)\}$ represent the bisector between sites s_i and s_j.

Definition 2.1. If w_R is the exact weight of R when two red cells of $\mathrm{VD}^w(S)$ meet for the first time at point c, then c is a *critical point of type I* if there exist red sites r_i and r_j and blue sites b_k and b_l such that $\{c\} = b(r_i, b_k) \cap b(r_i, b_l) \cap b(r_j, b_k) \cap b(r_j, b_l)$. Otherwise c is a *type II critical point* if $\{c\} = \overrightarrow{b_i r_i} \cap b(r_i, r_j)$, where $\overrightarrow{b_i r_i}$ is a ray from b_i to r_i.

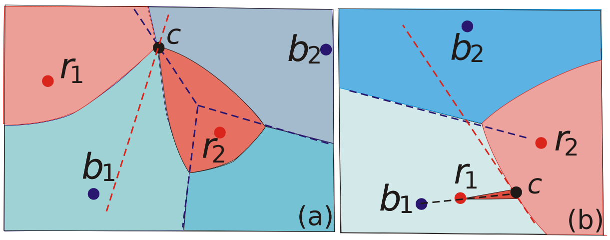

FIGURE 2. $\mathrm{VD}(R)$ is shown in a dashed red trace and $\mathrm{VD}(B)$ in dark blue. Point c is a critical point of type I in (a) and of type II in (b).

As before, type I critical points are found on the intersections of $\mathrm{VD}(R)$ and $\mathrm{VD}(B)$ (see Figure 2(a)). A type II critical point $\{c\} = \overrightarrow{b_1 r_1} \cap b(r_1, r_2)$ is shown in Figure 2(b).

Proposition 2.2. *Red connected components of* $\mathrm{VD}^w(S)$ *can only meet at critical points of type I or II, and they will meet* $\mathcal{O}(n)$ *times at most.*

Proposition 2.3. *There are* $\mathcal{O}(nm)$ *candidates to critical points.*

Once again, there is a gap between the actual number of critical points and the number of candidates to critical points. As soon as the list of these candidates is found, w_R^* can be located by means of a binary search. Since this is the same method as used in Theorem 1.5, the proof of the ensuing result is omitted.

Theorem 2.4. *Concerning the additively-weighted distance, the weight* w_R^* *can be found in* $\mathcal{O}(nm \log(nm))$ *time.*

Observe that it is possible for a type II critical point to exist "in the infinity" and this is true for both weighted distances. Nonetheless, this solution is not interesting to our problem as a path visiting all red cells would be infinitely long. Therefore, we assume the existence of a bounding box or of a polygon containing all the sites of S. In both cases, this type of critical point can be found as the intersection between the respective bisector (between the red sites) and such bounding box.

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